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Conformal analogue of the Adler–Gel'fand–Dikii bracket in two dimensions

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Abstract. In this paper we classify all differential invariants of parametrized curves in $\mathbb{R}^2$ under the action of O(3, 1). We find a formula for the most general evolution of such curves which are invariant under the action. We show that our formula induces a natural evolution on a generating set of differential invariants and we prove that such an evolution is Hamiltonian, giving an explicit expression of its Poisson tensor.

1. Introduction

The second Hamiltonian structure for KdV equations or Adler–Gel'fand–Dikii (AGD) brackets were defined originally by Adler [1] in order to prove that generalized KdV evolutions were infinite-dimensional biHamiltonian systems and hence completely integrable. This bracket was presumed to be Poisson until Gel'fand and Dikii proved its Jacobi identity in a real *tour de force* found in [7]. Since the original definition due to Adler was not very intuitive, and the Jacobi identity certainly not trivial, several alternative definitions have been given since then by different authors [13], [3] and [15], in an effort to understand their true significance.

The original definition was algebraic in nature, as was the definition of Kuperschmidt and Wilson that followed it. The interpretations of these brackets became increasingly geometric and one of them, perhaps the most striking one given by Drinfeld and Sokolov in [3], allowed their generalization from their original GL(n) and SL(n) cases, to the case of any semisimple Lie algebra $\mathfrak{g}$. In the case of SL(n), the author recently gave a purely geometrical definition in [15]. It could be summarized as follows.

Consider $I_1, I_2, \ldots, I_n$ to be *projective differential invariants* of parametrized curves in $\mathbb{R}^n$ generating any other invariant (that is, any other differential invariant will be a combination of them and their derivatives with respect to the curve parameter. For example, in the case of Euclidean plane geometry, the curvature of the curve would be the generating differential invariant. For further explanations see the following section). Since I_j are given explicitly in terms of the coordinates of the curve and their derivatives, they can be thought of as maps from the jet space J^k of parametrized curves in $\mathbb{R}^n$ to $\mathbb{R}$, which are invariant under the projective *prolonged action* of SL(n) on J^k. k is a certain appropriate order (see the next section for the precise definition). Now, let

$$\phi_t = F(\phi, \phi_\theta, \phi_{\theta\theta}, \ldots) \tag{1.1}$$

be the most general evolution of curves in $\mathbb{R}^n$ which is *invariant* under the projective action of SL(n) on $\mathbb{R}^n$ (that is, the group takes solutions to solutions). This evolution naturally induces an evolution on the variables of the jet space via differentiation, and hence an evolution on the invariants I_j, $j = 1, 2, \ldots, n$. In [15] it was proved that, under certain rather natural assumptions, this evolution is given by the AGD bracket. Moreover, vice versa, any AGD evolution can be viewed as an invariant evolution of differential invariants. Hence, the AGD evolution can be defined as the general invariant evolution of projective differential invariants. This approach also allows us to reinterpret the Poisson tensor as what is usually called a *relative differential invariant* of the projective action of $\mathrm{SL}(n)$. The geometric significance of the Jacobi identity in this invariant setting is still unknown. See [8] and [9] for a physical motivation to the relationship between evolutions of differential invariants and KdV.

The aim of this paper is to advance the study of infinite-dimensional Hamiltonian evolutions related to conformal geometry. That is, we would like to know if there is an analogue of the AGD Poisson bracket when we consider the conformal action of $\mathrm{O}(n + 1, 1)$ on $\mathbb{R}^n$ instead of the action of $\mathrm{SL}(n + 1)$. In particular, we study the $\mathrm{O}(3, 1)$ case.

Approaching Hamiltonian evolutions in infinite dimensions from an invariant point of view presents a fundamental problem; namely, the differential invariants of many of the main groups are not completely classified, including many of the groups of physical importance (general linear, affine, conformal etc). Therefore, if one follows this approach, the search for independent and generating differential invariants becomes part of the search for new Poisson tensors in infinite dimensions. On the other hand, this setting allows us the study of unexpected generalizations, for example the case of Poisson tensors in several independent variables. Indeed, the present author showed in [14] that differential invariants of reparametrizations of the projective plane produce, via their invariant evolution, a family of KdV Poisson structures in *every direction* of the projective plane. She also explained how KP equations are not likely to be found through this approach due to its non-local character.

The subject of differential invariants and invariant evolutions is in itself highly interesting. Among its applications, the study and classification of differential invariants and invariant evolutions of curves and surfaces under certain groups (Euclidean, affine, projective and conformal) has become relevant in the subject of image enhancing and image processing (see [20] and references within). The classification of differential invariants of reparametrizations of $\mathbb{R}^n$ or parametrized submanifolds is directly related to the classification of invariant cocycles of some infinite-dimensional Lie algebras, which is itself related to quantization. There is, of course, a traditional interest in invariants since they allow us to classify submanifolds up to the action of the group. That is, if two submanifolds ought to be equivalent (the group takes one to the other) they must have the same invariants. A traditional method to find these invariants is *the method of moving frames* described by Cartan. A modification of Cartan's method has been recently developed by Fels and Olver in [4] and [5]. The new method is more practical in many instances, especially if one needs to find explicit expressions for the invariants. Most notably, the method of Fels and Olver bypasses the complications that one faces in the process of normalization found in the traditional approach. We will make use of this new method in the next section.

The paper is divided into three additional sections.

In section 2, using the method of Fels and Olver, I find a complete set of differential invariants of parametrized curves in $\mathbb{R}^2$, invariant under the *prolonged* conformal action of $\mathrm{O}(3, 1)$. I show that these invariants are independent and generating. To the extent of my knowledge, they are not previously known.

In section 3 I find a nondegenerate matrix of *relative invariants* and, using that matrix, I find a formula for the most general evolution of parametrized curves in $\mathbb{R}^2$, invariant under the conformal action of $\mathrm{O}(3, 1)$.

Finally, in section 4 I calculate the evolution induced upon the invariants by the invariant evolution found in section 3. I show that such an evolution is Hamiltonian. I could not find any reference to this special infinite-dimensional Poisson bracket in the literature, although the literature on this subject is so enormous that it could be known. It would certainly be very interesting to relate this tensor to some known evolution. Since in dimension $n = 1$ (the action of O(2, 1) and SL(2) on $\mathbb{R}$) conformal and projective geometries coincide, both O(2, 1) and SL(2) will produce the usual second KdV structure, or the Lie–Poisson bracket on the dual of the Virasoro algebra. On the other hand, the next KdV Poisson structure in the AGD family (the case of SL(3) acting on $\mathbb{R}^2$ which produces a system with two equations) is certainly different from the bracket found here. Thus, the new Poisson bracket presented in this paper can certainly be called the conformal AGD bracket on the plane, or the conformal generalization of the Lie–Poisson bracket on the dual of the Virasoro algebra (as opposed to the usual AGD bracket, which is its projective generalization). It is still not clear to me whether this bracket has any relationship with the Drinfeld and Sokolov bracket, which is defined via a Poisson reduction on the dual of a Kac–Moody algebra, or whether this is an entirely different branch.

I close the paper with remarks about higher-dimensional conformal cases (i.e. is there a possible conformal family of brackets in view, one for each dimension n?), the study of surfaces, and other possible paths of study that this result opens, some of them already in progress.

This paper tries to be self-contained. Since the method of Fels and Olver is new, we present a summary adapted to our particular study. It is also not necessary for the reader to have knowledge about the projective case and AGD brackets, except perhaps to understand the reasons why I chose to study the invariant theory of the O(3, 1) conformal action on $\mathbb{R}^2$ and its associated invariant evolutions.

2. Conformal invariants of parametrized curves in $\mathbb{R}^2$

In this section we classify all differential invariants of parametrized curves in $\mathbb{R}^2$, invariant under the conformal action of O(3, 1). For this, we find a set of generating and independent differential invariants. But first, I will give some definitions and known results that will create the foundation for the study. For more information in the subject see [18] or [16].

2.1. Differential invariants

We begin by reviewing the basic theory of prolonged transformation groups and differential invariants. Let M be an m-dimensional manifold. We shall consider p-dimensional submanifolds parametrized by immersions $\iota: X \to M$, where X is a fixed parameter space, which, since we are only interested in local issues, can be taken to be an open subset of $\mathbb{R}^p$.

Let G be an r-dimensional Lie group acting smoothly on M. In particular, since we will study the case of parametrized curves, *we are assuming that G does not act on the parameters* $x \in X$. Let $G_S = \{g \in G \mid g \cdot S = S\}$ denote the *isotropy subgroup* of a subset $S \subset M$, and $G_S^* = \bigcap_{x\in S} G_x$ its *global isotropy subgroup*. We assume that G acts *effectively on subsets* of M, which means that $G_U^* = \{e\}$ for every open $U \subset M$. If an analytic transformation group acts effectively, it automatically acts effectively on subsets, but this equivalence does not hold in the smooth category. We say that G acts freely if $G_u = \{e\}$ for all $u \in M$. We further incorporate the adjective 'locally' in these concepts by replacing $\{e\}$ by a general discrete subgroup of G.

Let $\mathrm{J}^n = \mathrm{J}^n(X, M)$ denote the nth-order jet bundle consisting of equivalence classes of submanifolds modulo nth-order contact. We introduce local coordinates $x = (x^1, \ldots, x^p)$ on

X, and $u = (u^1, \ldots, u^q)$ on M. The induced local coordinates on J^n are denoted by $u^{(n)}$, with components u^α_J, where $J = (j_1, \ldots, j_k)$, $1 \leqslant j_\nu \leqslant p$, $0 \leqslant k \leqslant n$, $\alpha = 1, \ldots, q$, representing the partial derivatives of the dependent variables (u) with respect to the independent variables (x). Note that

$$\dim \mathrm{J}^n = q^{(n)} = q\binom{p+n}{n}. \tag{2.1}$$

Since G preserves the order of contact between submanifolds, there is an induced action of G on the jet bundle J^n known as its nth *prolongation*, and denoted by $G^{(n)}$ (the underlying group being identical to G). Since we are studying the action on parametrized submanifolds, G does not act on x, and the prolonged action becomes quite simple, namely the action is given by

$$G^{(n)} \times \mathrm{J}^n \to \mathrm{J}^n$$
$$(g, u_J) \to (g \cdot u)_J.$$

In this paper we need to consider the action on parametrized curves, versus intrinsic differential invariants, if we hope to establish any relationship between invariant evolutions and Hamiltonian systems. The later ones are essentially PDEs and the group acts on them taking solutions to solutions. These solutions have a parameter which remains unchanged under this action, so if we want the differential invariants to be solutions of these Hamiltonian evolutions we need to consider the action on parametrized curves, the parameter also being invariant.

Definition 2.1. *An nth-order differential invariant is a function $I\colon \mathrm{J}^n \to \mathbb{R}$ which is invariant under the action of $G^{(n)}$.*

Let s_n denote the maximal orbit dimension of the prolonged action $G^{(n)}$ on J^n. The *stable orbit dimension* is $s = \max s_n$. The *stabilization order* of G is the minimal n such that $s_n = s$. The *regular subset* $\mathcal{V}^n \subset \mathrm{J}^n$ is the open subset consisting of all prolonged group orbits of dimension equal to the stable orbit dimension, while the *singular subset* is $\mathcal{S}n = \mathrm{J}^n \setminus \mathcal{V}^n$. Note that, by this definition, $\mathcal{V}^n = \emptyset$ and $\mathcal{S}n = \mathrm{J}^n$ if n is less than the stabilization order of G. If G acts locally effectively on subsets, then the stabilization theorem, [18, 19], states that $s = r = \dim G$, which means that $G^{(n)}$ acts locally freely on $\mathcal{V}^n$ for all n.

Proposition 2.2. *In a neighbourhood of any regular jet $u^{(n)} \in \mathcal{V}^n$, there exist $q^{(n)} - s$ functionally independent differential invariants of order at most n.*

The traditional method for computing higher-order differential invariants is via the method of invariant differentiation. In the present situation, since G does not transform the parameters, the invariant differential operators are particularly simple. Namely, the parametric total derivative operators $D_i = D_{x^i}$, $i = 1, \ldots, p$, map differential invariants to differential invariants.

Proposition 2.3. *If $I(u^{(n)})$ is any differential invariant, so are its total derivatives $D_J I = D_{j_1} \cdots D_{j_k} I$, where $1 \leqslant j_\nu \leqslant p$, $k = \#J \geqslant 0$.*

Definition 2.4. *A generating set of differential invariants is a finite collection $I_1, \ldots, I_N$ with the property that, for all n, every differential invariant (on an appropriate subset of $\mathcal{V}^n$) can be written as a function of the derivatives $D_J I_\nu$ of the generating differential invariants.*

Example 2.5. Consider the simple case of the SL(2) action on $\mathbb{R}$ given by fractional transformations

$$\mathrm{SL}(2) \times \mathbb{R} \to \mathbb{R}$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, u\right) \to \frac{au+b}{cu+d}.$$

Consider curves u: $\mathbb{R} \to \mathbb{R}$ (to be precise, reparametrizations of $\mathbb{R}$), so that we have one dependent variable u and one independent variable x. The prolonged action of SL(2) on $\mathrm{J}^n(\mathbb{R}, \mathbb{R})$ is given by

$$\mathrm{SL}(2)^{(n)} \times \mathrm{J}^n(\mathbb{R}, \mathbb{R}) \to \mathrm{J}^n(\mathbb{R}, \mathbb{R})$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, u^{(k)}\right) \to \left(\frac{au + b}{cu + d}\right)^{(k)}$$

where (k) represents here the kth derivative with respect to x. One can show that the condition $u' \neq 0$ defines a regular jet and that the stabilization order is 2. In this case, there exists one generating differential invariant, namely the Schwarzian derivative of u

$$S(u) = \frac{u'''}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2.$$

Any other differential invariant will be a function of $S(u)$ and its derivatives with respect to x.

Every transformation group admits a generating system of differential invariants. The order of a generating system can be taken to be $n + 1$, where n is the stabilization order. The minimal order and minimal number of differential invariants required to form a generating system is not known in general, except in the particular case of curves, $p = 1$, [18].

Each of the preceding constructions has an infinitesimal counterpart. We choose a basis

$$\boldsymbol{v}_\kappa = \sum_{\alpha=1}^{q} \varphi_\kappa^\alpha(u) \frac{\partial}{\partial u^\alpha} \qquad \kappa = 1, \ldots, r \tag{2.2}$$

for the Lie algebra $\mathfrak{g}$ of infinitesimal generators on M. Let $\{\mathrm{pr}^{(n)}\, \boldsymbol{v}_1, \ldots, \mathrm{pr}^{(n)}\, \boldsymbol{v}_r\}$ denote the corresponding infinitesimal generators for the prolonged group action $G^{(n)}$. The prolonged generator $\mathrm{pr}^{(n)}\, \boldsymbol{v}_\kappa$ is obtained by truncating the infinitely prolonged vector fields, which in the special case of parametrized submanifolds takes the form

$$\mathrm{pr}\, \boldsymbol{v}_\kappa = \textstyle\sum_{\alpha=1}^{q} \sum_{k=\#J\geqslant 0} D_J \varphi_\kappa^\alpha(u^{(k)}) \frac{\partial}{\partial u_J^\alpha} \tag{2.3}$$

at order n. The dimension of the orbit passing through $u^{(n)} \in \mathrm{J}^n$ equals the dimension of the subspace of $T\mathrm{J}^n|_{u^{(n)}}$ spanned by $\mathrm{pr}^{(n)}\, \boldsymbol{v}_1, \ldots, \mathrm{pr}^{(n)}\, \boldsymbol{v}_r$. In particular, if G acts effectively on subsets, *a jet $u^{(n)}$ is regular if and only if the vector fields* $\mathrm{pr}^{(n)}\, \boldsymbol{v}_1, \ldots, \mathrm{pr}^{(n)}\, \boldsymbol{v}_r$ *are linearly independent there* (we will make use of this fact in the following sections). The infinitesimal invariance criteria are standard [18].

Proposition 2.6. *A function I: $\mathrm{J}^n \to \mathbb{R}$ is a differential invariant if and only if it is annihilated by the infinitesimal generators:* $\mathrm{pr}\, \boldsymbol{v}_\kappa(I) = 0$, $\kappa = 1, \ldots, r$.

2.2. The regularized moving frame method

We now describe how to implement the regularized moving frame method found in [4] and [5] in the particular case of parametrized submanifolds of an m-dimensional manifold M. Let $g = (g^1, \ldots, g^r)$ be local coordinates on G in a neighbourhood of the identity. Let us write out the group transformations $v = g \cdot u$ in local coordinates

$$v^\alpha = \Phi^\alpha(g^1, \ldots, g^r, u^1, \ldots, u^m). \tag{2.4}$$

The functions v^α in (2.4) are referred to as the zeroth-order lifted invariants, since they are invariant under the simultaneous action $(h, u) \mapsto (h \cdot g^{-1}, g \cdot u)$ of G on the trivial principal bundle $G \times M$. Since G does not act on the parameters, the corresponding prolonged transformations $v^{(n)} = g^{(n)} \cdot u^{(n)}$ are easily obtained by total differentiation. The resulting functions $v^\alpha_J = D_J v^\alpha$ are called the lifted differential invariants since they are invariant under the simultaneous action $(h, u^{(n)}) \to (h \cdot g^{-1}, g^{(n)} \cdot u^{(n)})$ on $G \times \mathrm{J}^n$.

The primary use of a moving frame is that it enables one to pass from lifted invariant objects, which are trivial, to their ordinary invariant counterparts back on the original manifold and its jet spaces. This allows us to systematically analyse the invariants via the particularities of the moving frame. The following fundamental definition appears in [5], and is motivated by earlier work of Griffiths [11] and Jensen [12].

Definition 2.7. *An nth-order moving frame is a map $\rho^{(n)}: \mathrm{J}^n \to G$ which is (locally) G-equivariant with respect to the prolonged action $G^{(n)}$ on J^n, and the right action $h \mapsto h \cdot g^{-1}$ of G on itself.*

Remark. For simplicity, we shall only consider right-moving frames in this paper. A left-moving frame, which is equivariant with respect to left multiplication $h \mapsto g \cdot h$ is easily obtained by inverting the right-moving frame: $\rho^{(n)}(g)^{-1}$.

Theorem 2.8. *If G acts effectively on subsets, then an nth-order moving frame exists in a neighbourhood of a point $u^{(n)} \in \mathrm{J}^n$ if and only if $u^{(n)} \in \mathcal{V}^n$ is a regular jet.*

In particular, the minimal order at which any moving frame exists is the stabilization order of the group. In practical implementations, the normalization procedure for constructing moving frames amounts to choosing a (local) cross-section $\mathcal{K}^n \subset \mathcal{V}^n$ to the (regular) prolonged group orbits. In other words, $\mathcal{K}^n$ is a submanifold of dimension $q^{(n)} - r$ which intersects each orbit at most once, and transversally. Given $u^{(n)} \in \mathcal{V}^n$, let $g = \rho^{(n)}(u^{(n)})$ denote the group element that maps $u^{(n)}$ to the cross-section:

$$g^{(n)} \cdot u^{(n)} = \rho^{(n)}(u^{(n)}) \cdot u^{(n)} \in \mathcal{K}^n. \tag{2.5}$$

The resulting map $\rho^{(n)}: \mathrm{J}^n \to G$ is a moving frame. Moreover, every moving frame has this form, where the cross-section equals the preimage $\mathcal{K}^n = (\rho^{(n)})^{-1}\{e\}$ of the identity element.

The simplest local cross-sections are obtained by setting $r = \dim G$ of the jet coordinates $u^{(n)}$ to be constant. We denote the chosen coordinates by $u_\nu \equiv u^{\alpha_\nu}_{J_\nu}$, $\nu = 1, \ldots, r$. Therefore, $\mathcal{K}^n = \{u_1 = c_1, \ldots, u_r = c_r\}$, where the normalization constants $c_1, \ldots, c_r$ are chosen so that the *normalization equations* (2.5), which have the form

$$v_1 = v^{\alpha_1}_{J_1}(g, u^{(n)}) = c_1 \qquad \ldots \qquad v_r = v^{\alpha_r}_{J_r}(g, u^{(n)}) = c_r \tag{2.6}$$

can be (locally) uniquely solved for $g = \rho^{(n)}(u^{(n)})$ in terms of the jet coordinates. The resulting map defines the moving frame associated with the chosen cross-section.

Remark. Any nth-order moving frame $\rho^{(n)}: \mathrm{J}^n \to G$ can also be viewed as a moving frame of any higher-order $k \geqslant n$ by composing with the standard jet space projection $\pi^k_n: \mathrm{J}^n \to \mathrm{J}^k$. In the sequel, we will speak of 'moving frames of order n' with the understanding that they may very well have been constructed at some lower order.

Example 2.9. In the SL(2) case considered before, a moving frame at a regular jet with $u' > 0$ is obtained using the cross-section

$$\mathcal{K}^2 = \{u = 0, u' = 1, u'' = 0\}.$$

The normalization equations are thus given by

$$v = \frac{au + b}{cu + d} = 0 \qquad v' = \frac{u'}{(cu + d)^2} = 1 \qquad v'' = \frac{u''}{(cu + d)^2} - \frac{2cu'^2}{(cu + d)^3} = 0.$$

Solving the normalization equations results in the second-order moving frame

$$\rho^{(2)} : \mathrm{J}^2(\mathbb{R}, \mathbb{R}) \to \mathrm{SL}(2)$$
$$(u, u', u'') \to \begin{pmatrix} u'^{-\frac{1}{2}} & uu'^{-\frac{1}{2}} \\ \frac{1}{2}\frac{u''}{u'^{\frac{3}{2}}} & u'^{\frac{1}{2}} - \frac{1}{2}u\frac{u''}{u'^{\frac{3}{2}}} \end{pmatrix}.$$

We now describe how the moving frame provides us with a complete system of differential invariants.

Definition 2.10. *The fundamental nth-order normalized differential invariants associated with a moving frame $\rho^{(n)}$ of order n or less are given by*

$$I^{(n)}(u^{(n)}) = v^{(n)}(\rho^{(n)}(u^{(n)}), u^{(n)}) = \rho^{(n)}(u^{(n)}) \cdot u^{(n)}. \tag{2.7}$$

In other words, the individual components of $I^{(n)}$, which are

$$I_K^\alpha(u^{(k)}) = v_K^\alpha(\rho^{(n)}(u^{(n)}), u^{(k)}) \qquad \alpha = 1, \ldots, q \qquad k = \#K \geqslant 0 \tag{2.8}$$

define differential invariants of order $\leqslant n$. Note that the normalized differential invariants corresponding to the components being normalized via (2.6) will be constant. We shall call these the *phantom differential invariants*. The other components of $v^{(n)}$ will define a complete system of functionally independent differential invariants defined on the domain of definition of the moving frame map. This will hold for any order n at least as large as the order of the chosen moving frame.

Theorem 2.11. *Let n be greater than or equal to the order of the moving frame. Every nth-order differential invariant can be locally written as a function of the normalized nth-order differential invariants $I^{(n)}$. The function is unique provided it does not depend on the phantom invariants.*

Example 2.12. In the SL(2) case, the second-order differential invariants are all constant (they are phantom invariants), but if for simplicity we denote $\rho^{(2)}(u^{(2)})$, the frame we found in example 2.9, by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the third-order differential invariant is given by

$$\rho^{(2)}(u^{(2)}) \cdot u^{(3)} = \frac{u'''}{(cu+d)^2} - 6\frac{cu'u''}{(cu+d)^3} + 6\frac{c^2u'^3}{(cu+d)^4} = \frac{u'''}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2 = S(u).$$

Notice that, given the definition of prolonged action, the invariant is obtained finding $(g \cdot u)'''$ in general first, and substituting the moving frame in the resulting expression *after differentiation*.

A moving frame therefore provides a natural way to construct a differential invariant from any differential function.

Definition 2.13. *The invariantization with respect to the given moving frame of a differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ is the differential invariant $F \circ I^{(n)}$.*

In particular, if F is itself a differential invariant, then it coincides with its invariantization: $F = F \circ I^{(n)}$. Thus, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants.

An alternative method to construct higher-order differential invariants is by invariant differentiation, as in proposition 2.3. A critical remark, however, is that the total derivative of a normalized differential invariant is *not* necessarily equal to the corresponding higher-order normalized differential invariant. The fundamental *recurrence formulae* for the differential invariants (2.8) are

$$D_j I_K^\alpha = I_{K,j}^\alpha + M_{K;j}^\alpha. \tag{2.9}$$

Higher-order versions,

$$D_J I^\alpha_K = I^\alpha_{K,J} + M^\alpha_{K;J} \tag{2.10}$$

are obtained by differentiating (2.9). For example,

$$D_k D_j I^\alpha_K = D_k I^\alpha_{K,j} + D_k M^\alpha_{K;j} = I^\alpha_{K,j,k} + \sum_{\beta,L} \frac{\partial M^\alpha_{K;j}}{\partial I^\beta_L}(I^\beta_{L,k} + M^\beta_{L;k}).$$

Remark. While $I^\alpha_{K,J}$ is symmetric under permutations of the multi-index (K, J), this is not true for $M^\alpha_{K;J}$, which is why we use a semicolon to separate the two indices.

The 'correction terms' $M^\alpha_{K;j}$ can be explicitly computed using a certain algorithm, which can be found in [6]. As we will see below, except in the case of curves, where $p = 1$, the differentiated invariants are not necessarily functionally independent. A *syzygy* is a functional dependency $H(\ldots \mathcal{D}_J I_\nu \ldots) \equiv 0$ among the fundamental differentiated invariants. The recurrence formulae not only provide us with a generating system of fundamental differential invariants, but also classifies all syzygies among the normalized differential invariants.

Theorem 2.14. *A generating system of differential invariants consists of (a) all non-phantom zeroth-order differential invariants I^α, and (b) all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant. In other words, every other differential invariant can, locally, be written as a function of the generating invariants and their invariant derivatives, $D_K I^\alpha_{J,i}$.*

All syzygies among the differentiated invariants are differential consequences of the following two fundamental types:

(i) $D_J I^\alpha_K = c_\nu + M^\alpha_{K,J}$, when I^α_K is a generating differential invariant, while $I^\alpha_{J,K} = c_\nu$ is a phantom differential invariant, and

(ii) $D_J I^\alpha_{LK} - D_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}$, where I^α_{LK} and I^α_{LJ} are generating differential invariants, the multi-indices $K \cap J = \emptyset$ are disjoint and non-zero, while L is an arbitrary multi-index.

A minimal system of differential invariants can be found by a careful analysis of the recurrence relations and consequent syzygies. Examples appear in [5] and [16].

2.3. A regularized moving frame for conformal curves in $\mathbb{R}^2$

In this section we will apply the regularized moving frame method to our particular case of curves in $\mathbb{R}^2$, under the conformal action of O(3, 1). The study will follow the following outline: after describing the action, we will find regular prolonged orbits. In order to find a moving frame along regular jets we need to choose a local cross-section to regular prolonged group orbits. This is accomplished via the normalization of a certain number of jet coordinates. The group element taking a nearby general element $u^{(n)}$ to the cross-section defines the moving frame. The normalization equations (2.6) obtained when normalizing the corresponding lifted differential invariants give us, when solved, the explicit expression for the moving frame. We finally use the frame to find a generating system of differential invariants, via theorem 2.14.

In our particular case, $X = \mathbb{R}$ or an open subset of $\mathbb{R}$, $M = \mathbb{R}^2$ and $G = \mathrm{O}(3, 1)$, acting on $\mathbb{R}^2$ conformally. That is, the action is given locally by

$$\begin{aligned}
&\mathrm{O}(3,1) \times \mathbb{R}^2 \to \mathbb{R}^2\\
&((a^i_j), (u^1, u^2)) \to (v^1, v^2)\\
&v^1 = \frac{a^2_1 q + a^2_2 u^1 + a^2_3 u^2 + a^2_4}{a^4_1 q + a^4_2 u^1 + a^4_3 u^2 + a^4_4} \qquad v^2 = \frac{a^3_1 q + a^3_2 u^1 + a^3_3 u^2 + a^3_4}{a^4_1 q + a^4_2 u^1 + a^4_3 u^2 + a^4_4}
\end{aligned} \tag{2.11}$$

where q is such that $2q + (u^1)^2 + (u^2)^2 = 1$. The conformal action of $\mathrm{O}(n+1, 1)$ on $\mathbb{R}^n$ can be described as follows.

Let $\mathrm{O}(n+1, 1)$ be the set of matrices preserving the indefinite Minkowski metric defined via the matrix

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

That is, $N \in \mathrm{O}(n+1, 1)$ if, and only if, $NCN^T = C$, where ^{T} denotes transposition.

Let $\mathbb{RP}_0^{n+1}$ be the lightcone in Minkowski space, that is, points in $\mathbb{RP}^{n+1}$ with zero Minkowski length. We can also think of them as lines in $\mathbb{R}^{n+2}$ such that $xCx^T = 0$ whenever x is on the line.

$\mathrm{O}(n+1, 1)$ acts naturally on $\mathbb{R}^{n+2}$ via the usual multiplication. Given that $\mathrm{O}(n+1, 1)$ preserves the metric, it also acts on $\mathbb{RP}_0^{n+1}$ if we immerse $\mathbb{RP}^{n+1}$ into $\mathbb{R}^{n+2}$ the usual way. If $U \in \mathbb{RP}^{n+1}$ is a coordinate chart, the immersion of $\mathbb{RP}^{n+1}$ into $\mathbb{R}^{n+2}$ will take locally the form

$$\begin{aligned} \eta : U &\to \mathbb{R}^{n+2} \\ y &\to (y, 1). \end{aligned}$$

Now, $\mathbb{R}^n$ can be identified locally with $\mathbb{RP}_0^{n+1}$ using the map

$$\begin{aligned} \nu : \mathbb{R}^n &\to \mathbb{RP}_0^{n+1} \\ u &\to (q, u) \end{aligned}$$

where q is uniquely determined through the relationship $2q + (u^1)^2 + \cdots + (u^n)^2 = 0$ which is necessary upon imposing the zero length condition. Here $u = (u^1, \ldots, u^n)$. Let π be the projection of $\mathbb{R}^{n+2} - \{0\}$ on $\mathbb{RP}^{n+1}$.

The action of $\mathrm{O}(n+1, 1)$ on $\mathbb{R}^n$ is given by the map

$$\begin{aligned} &\mathrm{O}(n+1, 1) \times \mathbb{R}^n \to \mathbb{R}^n \\ &(N, u) \to N \cdot u = \nu^{-1}(\pi N(\eta(\nu(u)))) \end{aligned} \tag{2.12}$$

that is, we lift u to an unique element on the lightcone, lift the line to $\mathbb{R}^{n+2}$, multiply by N and project back into $\mathbb{RP}_0^{n+1}$ and into $\mathbb{R}^n$. In the case $n = 2$, this procedure results in (2.11).

Since a general element $g = (a^i_j) \in \mathrm{O}(3, 1)$ preserves the indefinite conformal metric, its entries also hold some relationships. These are given by the equations

$$(a^i_2)^2 + (a^i_3)^2 + 2a^i_1 a^i_4 = 1 \qquad i = 2, 3 \tag{2.13}$$

$$(a^i_2)^2 + (a^i_3)^2 + 2a^i_1 a^i_4 = 0 \qquad i = 1, 4 \tag{2.14}$$

$$a^i_2 a^j_2 + a^i_3 a^j_3 + a^i_1 a^j_4 + a^i_4 a^j_1 = 0 \qquad i, j = 1, \ldots, 4 \qquad i \neq j \tag{2.15}$$

which the entries of g must satisfy. They form a total of ten equations. We knew that number since the dimension of $\mathrm{O}(3, 1)$ is equal to six. Notice that the entries a^1_j, $j = 1, \ldots, 4$, do not appear in the formula of the action (2.11). They have a role up to the point where $\mathrm{O}(3, 1)$ acts on $\mathbb{RP}_0^3$, but they disappear upon projection. Hence, we are going to drop the four equations involving these entries from (2.13)–(2.15) with the certainty that they will not be needed to find the differential invariants. Of course, the moving frame itself will not be complete without them, but the interested reader can always find their explicit expression from the four equations we intend to ignore, after we have found the values of the remaining entries.

For convenience we will introduce the following notation.

Let $D = a_1^4 q + a_2^4 u^1 + a_3^4 u^2 + a_4^4$. Denote by A_j^i the following expressions: $A_2^i = a_2^i - u^1 a_1^i$, $A_3^i = a_3^i - u^2 a_1^i$, $A_4^i = a_4^i - q a_1^i$, for $i = 2, 3, 4$. With this new notation, the equations above (2.13)–(2.15) can be rewritten in a simpler way. They become

$$(A_2^j)^2 + (A_3^j)^2 = 1 \qquad i = 2, 3 \tag{2.16}$$
$$(A_2^4)^2 + (A_3^4)^2 = -2Da_1^4 \tag{2.17}$$
$$A_2^i A_2^4 + A_3^i A_3^4 + Da_1^i = 0 \qquad i = 2, 3 \tag{2.18}$$
$$A_2^2 A_2^3 + A_3^2 A_3^3 = 0. \tag{2.19}$$

The final system we need to solve will prove to be simpler than the previous one if we make use of A_j^i instead of a_j^i.

Proposition 2.15. *Let $|c_t|$ be the (Euclidean) length of $c_t = (u_t^1, u_t^2)$, where t is the curve parameter and the subindex indicates differentiation. If $|c_t| \neq 0$, then $u^{(n)} \in \mathrm{J}^n$ is a regular jet for the prolonged action of* O(3, 1)*, for any $n \geqslant 2$.*

Proof. As we pointed out before proposition 2.6, it suffices to show that the vectors $\{\mathrm{pr}^{(n)} \boldsymbol{v}_1, \ldots, \mathrm{pr}^{(n)} \boldsymbol{v}_6\}$ are independent whenever $n \geqslant 2$, where the vectors $\boldsymbol{v}$ are infinitesimal generators for the conformal action on $\mathbb{R}^2$. It is known (see [18]) that a set of generators for the infinitesimal conformal action of o(3, 1) on $\mathbb{R}^2$ is given by the vectors

$$\begin{aligned} &\boldsymbol{v}_1 = \frac{\partial}{\partial u^1} \qquad \boldsymbol{v}_2 = \frac{\partial}{\partial u^2} \\ &\boldsymbol{v}_3 = u^1 \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^2} \qquad \boldsymbol{v}_4 = u^2 \frac{\partial}{\partial u^1} - u^1 \frac{\partial}{\partial u^2} \\ &\boldsymbol{v}_5 = ((u^1)^2 - (u^2)^2) \frac{\partial}{\partial u^1} + 2u^1u^2 \frac{\partial}{\partial u^2} \qquad \boldsymbol{v}_6 = 2u^1u^2 \frac{\partial}{\partial u^1} + ((u^2)^2 - (u^1)^2) \frac{\partial}{\partial u^2}. \end{aligned} \tag{2.20}$$

According to (2.3), if we denote by μ^{ij} the expression $\mu^{ij} = ((u^i)^2 - (u^j)^2)$, then the following matrix has the second-order prolongations of these six vectors as rows, in the obvious order:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ u^1 & u^2 & u_t^1 & u_t^2 & u_{tt}^1 & u_{tt}^2 \\ u^2 & -u^1 & u_t^2 & -u_t^1 & u_{tt}^2 & -u_{tt}^1 \\ \mu^{12} & 2u^1u^2 & \mu_t^{12} & 2(u^1u^2)_t & \mu_{tt}^{12} & 2(u^1u^2)_{tt} \\ 2u^1u^2 & \mu^{21} & 2(u^1u^2)_t & \mu_t^{21} & 2(u^1u^2)_{tt} & \mu_{tt}^{21} \end{pmatrix}.$$

Although it is a long and tedious calculation, it is fundamentally trivial to check that, after row reduction, the matrix above becomes the identity matrix, if we simply assume that $|c_t| \neq 0$ (and, hence, so is either u_t^1 or u_t^2). □

Given the previous result and the statement of theorem 2.8, we know that a moving frame of order two exists around any point of the curve such that $|c_t| \neq 0$. This frame can be found via the normalizations of a number of components of $(v^1, v^2)^{(k)}$ for $k \leqslant 2$.

Theorem 2.16. *A moving frame for the prolonged conformal action of* $\mathrm{O}(3, 1)^{(2)}$ *on* J^2 *is given by the following map:*

$$\begin{aligned} &\rho^{(2)} : \mathrm{J}^2 \to \mathrm{O}(3, 1) \\ &(u^1, u^2)^{(2)} \to (a_j^i) \end{aligned}$$

where

$$a_1^2 = \mp_2 \frac{c_t \cdot c_{tt}}{|c_t|^3}$$

$$a_1^3 = \pm_1 \frac{1}{|c_t|^3} \begin{vmatrix} u_t^1 & u_t^2 \\ u_{tt}^1 & u_{tt}^2 \end{vmatrix} \mp_2 \frac{1}{|c_t|}$$

$$a_1^4 = \mp_2 \frac{1}{2|c_t|^3}[|c_t|^2 + |c_{tt}|^2] \pm_1 \frac{1}{|c_t|^3} \begin{vmatrix} u_t^1 & u_t^2 \\ u_{tt}^1 & u_{tt}^2 \end{vmatrix}$$

$$a_2^2 = u^1 a_1^2 \pm_2 \frac{u_t^1}{|c_t|} \qquad a_3^2 = u^2 a_1^2 \pm_2 \frac{u_t^2}{|c_t|} \qquad a_4^2 = q a_1^2 \mp_2 \frac{c \cdot c_{tt}}{|c_t|}$$

$$a_2^3 = u^1 a_1^3 \mp_1 \frac{u_t^2}{|c_t|} \qquad a_3^3 = u^1 a_1^3 \mp_1 \frac{u_t^2}{|c_t|} \qquad a_4^3 = q a_1^3 \pm_1 \frac{1}{|c_t|} \begin{vmatrix} u^1 & u_t^1 \\ u^2 & u_t^2 \end{vmatrix}$$

$$a_2^4 = u^1 a_1^4 \pm_2 \frac{1}{|c_t|}[2u_t^1 u_t^2 u_{tt}^2 + ((u_t^1)^2 - (u_t^2)^2) u_{tt}^1] \mp_1 \frac{u_t^2}{|c_t|}$$

$$a_3^4 = u^2 a_1^4 \pm_2 \frac{1}{|c_t|}[2u_t^1 u_t^2 u_{tt}^1 + ((u_t^2)^2 - (u_t^1)^2) u_{tt}^2] \pm_1 \frac{u_t^1}{|c_t|}$$

a_4^4 *is determined by the relationship*

$$D = q a_1^4 + u^1 a_2^4 + u^2 a_3^4 + a_4^4 = \pm_2 |c_t|$$

and where a_i^1 *are determined by the four equations we have chosen to ignore. The signs* $\pm_1$ *and* $\pm_2$ *correspond to local choices and are independent of each other.*

Proof. Following the technique described by Fels and Olver, we need to specify the normalizations of six components of the six that form $(v^1, v^2)^{(2)}$ (that is, all of them). We also need to further choose the normalization constants so that equations (2.6) are uniquely solvable for the parameter a_j^i. The solution of that system will provide the element $\rho^{(2)} \in \mathrm{O}(3, 1)$ which takes (u^1, u^2) to the section transverse to the regular prolonged orbit, and which is itself prescribed by the normalizations chosen. $\rho^{(2)}$ defines the frame.

We choose the following normalizations

$$\begin{array}{lll} v^1 = 0 & v_t^1 = 1 & v_t^2 = 0 \\ v^2 = 0 & v_{tt}^1 = 0 & v_{tt}^2 = 1. \end{array} \tag{2.21}$$

Next, in the notation used for (2.16)–(2.19), (2.21) can be rewritten as

$$A_4^j + u^1 A_2^j + u^2 A_3^j = 0 \qquad j = 2, 3 \tag{2.22}$$

$$u_1^1 A_2^j + u_1^2 A_3^j = D\delta_j^2 \qquad j = 2, 3 \tag{2.23}$$

$$u_2^1 A_2^j + u_2^2 A_3^j - |c_t|^2 a_1^j = \begin{cases} 2D_t & j = 2 \\ D & j = 3 \end{cases} \tag{2.24}$$

where D_t is given by $D_t = a_1^4 q_t + a_2^4 u_t^1 + a_3^4 u_t^2$, and where δ_j^2 denotes the delta of Kronecker. Therefore, the frame will be found once we solve for the element $g = (a_j^i) \in \mathrm{O}(3, 1)$ using equations (2.16)–(2.19), and (2.22)–(2.24).

Indeed, using first (2.23) and (2.16) with $j = 3$ we obtain

$$A_2^3 = \mp_1 \frac{u_t^2}{|c_t|} \qquad A_3^3 = \pm_1 \frac{u_t^1}{|c_t|}. \tag{2.25}$$

Along these calculations we will introduce several possibilities for different signs (as a result of the nonlinear relationships imposed by the group condition). Hence we are numbering these signs as $\pm_{1,2,\ldots}$ to indicate the independence between those choices. Given the values we have obtained in (2.25), we can use (2.24) to obtain the value of a_1^3

$$a_1^3 = \frac{\pm_1}{|c_t|^3}\begin{vmatrix} u_t^1 & u_t^2 \\ u_{tt}^1 & u_{tt}^2 \end{vmatrix} - \frac{D}{|c_t|^2}. \tag{2.26}$$

We can next solve equations (2.16), (2.23) (for $j = 2$), together with (2.19) to obtain the values

$$A_2^2 = \pm_2 \frac{u_t^1}{|c_t|} \qquad A_3^2 = \pm_2 \frac{u_t^2}{|c_t|} \qquad D = \pm_2 |c_t|. \tag{2.27}$$

Making use of the remaining equations, we obtain the rest of the values, namely

$$a_1^2 = -\pm_2 \frac{1}{|c_t|^3} c_t \cdot c_{tt} \qquad D_t = \pm_2 \frac{1}{|c_t|} c_t \cdot c_{tt} \tag{2.28}$$

$$A_2^4 = \pm_2 \frac{1}{|c_t|}[2u_t^1 u_t^2 u_{tt}^2 + ((u_t^1)^2 - (u_t^2)^2)u_{tt}^1] \mp_1 \frac{u_t^2}{|c_t|} \tag{2.29}$$

$$A_3^4 = \pm_2 \frac{1}{|c_t|}[2u_t^1 u_t^2 u_{tt}^1 + ((u_t^2)^2 - (u_t^1)^2)u_{tt}^2] \pm_1 \frac{u_t^1}{|c_t|} \tag{2.30}$$

$$a_1^4 = \mp_2 \frac{1}{2|c_t|^3}[|c_t|^2 + |c_{tt}|^2] \pm_1 \frac{1}{|c_t|^3}\begin{vmatrix} u_t^1 & u_t^2 \\ u_{tt}^1 & u_{tt}^2 \end{vmatrix}. \tag{2.31}$$

Using the definition of A_j^i, we obtain the values of a_j^i appearing in the statement of the theorem. It is obvious that the group relations involving the parameters a_i^1, $i = 1, \ldots, 4$, as expressed in (2.14) and (2.15), determine the values of these parameters locally. $\square$

2.4. *A generating set of independent differential invariants*

Once we have a moving frame for the prolonged action of $\mathrm{O}(3, 1)$, obtaining a set of independent differential invariants is quite simple. Perhaps the most useful theorem for us is theorem 2.14, which describes not only a generating set of differential invariants, but also their syzygies—algebraic relations among the generating invariants and their derivatives.

Theorem 2.17. *A generating set of independent differential invariants for the conformal action of* $\mathrm{O}(3, 1)$ *on parametrized curves on the plane is given by the following two invariants:*

$$I^1 = \frac{c_t \wedge c_{ttt}}{|c_t|^2} - 3\frac{c_t \cdot c_{tt}}{|c_t|^4} c_t \wedge c_{tt} \tag{2.32}$$

$$I^2 = \frac{c_t \cdot c_{ttt}}{|c_t|^2} - \frac{3}{2}\frac{|c_{tt}|^2}{|c_t|^2} + 3\frac{(c_t \wedge c_{tt})^2}{|c_t|^4}. \tag{2.33}$$

Observe that, when calculating I^1 and I^2, the signs $\pm_1$ and $\pm_2$ are factored in the process from I^1 and I^2, respectively, and so the invariants do not depend on the choice of sign, as should be expected. Also, note that these invariants are in fact generalizations of the Schwarz derivative to the conformal case (the Schwarz derivative is the generating differential invariant for the projective action on reparametrizations of $\mathbb{RP}$, case $n = 1$).

Proof. Following theorem 2.14, we conclude that a generating system of differential invariants is given by the substitution of our frame in the expression for v_{ttt}^1 and v_{ttt}^2. That is, we need to find the expression for both of them *before normalizations* and normalize (i.e. apply the frame) *afterwards*. If we denote by N^i the numerator of v^i, that is, $N^i = a_1^{i+1} q + a_2^{i+1} u^1 + a_3^{i+1} u^2 + a_4^{i+1}$, then v_{ttt}^i can be written as

$$v_{ttt}^i = \frac{N_{ttt}^i}{D} - 3\frac{N_{tt}^i}{D}\frac{D_t}{D} + 3\frac{N_t^i}{D}\left(\frac{D_t}{D}\right)^2 - 3\frac{N_t^i}{D}\left(\frac{D_t}{D}\right)_t + R^i \tag{2.34}$$

where R^i depends on N^i, and will vanish after the normalizations $v^i = 0$, $i = 1, 2$. Next we apply normalizations (2.21). The first group, $v^i = 0$, implies $N^i = 0$, and hence $R^i = 0$. The second group, $v_t^1 = 1$, $v_t^2 = 0$, results in $N_t^1 = D$ and $N_t^2 = 0$. The third group, $v_{tt}^1 = 0$ and $v_{tt}^2 = 1$, gives us the values $N_{tt}^1 = 2D_t$ and $N_{tt}^2 = D$. With all these data substituted into (2.34) we obtain that I^2 and I^1 are given by the application of the frame to the expressions

$$\frac{N_{ttt}^1}{D} - 3\frac{D_{tt}}{D} \tag{2.35}$$

and

$$\frac{N_{ttt}^2}{D} - 3\frac{D_t}{D} \tag{2.36}$$

respectively. That is, I^2 and I^1 are produced by the above expressions *after normalization*. We can easily find $N_{ttt}^i = q_{ttt}a_1^{i+1} + u_{ttt}^1 a_2^{i+1} + u_{ttt}^2 a_3^{i+1}$ via the values (2.25)–(2.28), in addition to the relationship $q_{ttt} = -c \cdot c_{ttt} - 3c_t \cdot c_{tt}$. The values of D and D_t have also been directly obtained in (2.27) and (2.28). The only expression we need to solve for is D_{tt}, which is trivial since we know the value of the parameters a_1^4, a_2^4 and a_3^4 from the explicit expression of the frame. If we put together all these values in expressions (2.36) and (2.35) we obtain the values of I^1 and I^2 specified in the statement of the theorem.

The only point that needs further thought is the independence of these two invariants, but one can prove it easily making use of theorem 2.14. There we find a classification of all possible relationships between invariants and their derivatives (syzygies). As the theorem tell us they are all of the form

$$D_J I_K^\alpha = c_\nu + M_{K,J}^\alpha$$

and

$$D_J I_{LK}^\alpha - D_K I_{LJ}^\alpha = M_{LK,J}^\alpha - M_{LJ,K}^\alpha$$

where I^α are generating differential invariants or phantom invariants, depending on the equation, and where the differential multi-indices have certain conditions. What is important to us is the fact that *no two invariants obtained via normalizations of different lifted invariants* (above corresponding to different choices of α) *have any functional relationship among them or their derivatives that makes them dependent*. In our case, I^1 and I^2 are formed normalizing v_{ttt}^1 and v_{ttt}^2 and hence they must be independent. □

3. Evolutions in $\mathbb{R}^2$ which are conformally invariant

In this section we want to find a general formula for the evolution of curves in $\mathbb{R}^2$ of the form

$$u_s = F(u^1, u^2, u_t^1, u_t^2, u_{tt}^1, u_{tt}^2, \ldots) \tag{3.1}$$

which is invariant under the conformal action of $O(3,1)$ on $\mathbb{R}^2$; that is to say, if u is a solution of (3.1), we want $M \cdot u$ to be also a solution of (3.1), where M is any element in $O(3,1)$ and where $\cdot$ denotes the action of $O(3,1)$ on $\mathbb{R}^2$. In order to find such an evolution we introduce a definition and a known theorem about the character of what is called a *relative invariant*. The definition below is given in the general context found in earlier parts of section 2.

Definition 3.1. *Let F be an nth vector function defined on the jet space $F : \mathrm{J}^k \to \mathbb{C}^\infty(\mathbb{R}) \times \overset{n)}{\cdots} \times \mathbb{C}^\infty(\mathbb{R})$. Let $\boldsymbol{v}_\kappa = \sum_{\alpha=1}^q \varphi_\kappa^\alpha(u)\frac{\partial}{\partial u^\alpha}$, $\kappa = 1, \ldots, r$, be a generating set of infinitesimal generators of the action whose algebra of infinitesimal generators is given by $\mathfrak{g}$. Assume that*

$$\mathrm{pr}\ \boldsymbol{v}_\kappa(F) = \frac{\partial \varphi_\kappa}{\partial u} F \tag{3.2}$$

where $\frac{\partial \varphi_\kappa}{\partial u}$ *is the* $(n-1) \times (n-1)$ *matrix with* (i, j) *entry* $\frac{\partial \varphi_\kappa^i}{\partial u^j}$. *Then* F *is called a relative vector differential invariant of the Lie algebra* $\mathfrak{g}$, *whose associated weight is the matrix* $\frac{\partial \varphi}{\partial u}$.

One of the main properties of relative differential invariants is described in the following theorem, which can be found in [10].

Theorem 3.2. *The most general vector* F *making an* n*th dimensional evolution of the form (3.1) invariant under the action of a group* G *is given by*

$$F = \mu J$$

where the $n \times n$ *matrix* $\mu = (\mu^1\ \mu^2\ \cdots\ \mu^{n-1})$ *is any matrix with nonvanishing determinant and whose columns* μ^i *are particular solutions of (3.2), and where* $J = (J_k)_{k=1}^n$ *is an arbitrary* absolute (vector) differential invariant *of the algebra* $\mathfrak{g}$, *i.e. a solution of*

$$\mathrm{pr}\ \boldsymbol{v}(J_i) = 0 \qquad \textit{for all} \quad \boldsymbol{v} \in \mathfrak{g} \qquad i = 1, \ldots, n.$$

That is, in order to find a formula for the most general evolution of curves on the plane, invariant under the conformal action of O(3, 1), we need to find a nondegenerate matrix whose columns are relative differential invariants for the action.

Theorem 3.3. *Assume that* $c = (u^1, u^2)$ *holds* $|c_t| \neq 0$. *Then, any evolution of the form (3.1), invariant under the conformal action of* O(3, 1)*, can be written as*

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_s = \begin{pmatrix} u_t^1 & u_t^2 \\ -u_t^2 & u_t^1 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \tag{3.3}$$

where J_1 *and* J_2 *are any two differential invariants for the action. That is, they are functions of* I_1, I_2 *given in (2.32) and (2.33), and their derivatives with respect to* t.

Proof. In view of our previous discussion, it is clear that we simply need to show that the two vectors

$$\begin{pmatrix} u_t^1 \\ -u_t^2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} u_t^2 \\ u_t^1 \end{pmatrix}$$

are relative invariants for the conformal action of O(3, 1) on the plane. This is the result of a straightforward calculation using the set of infinitesimal generators given in (2.2), the formula for their prolongation given in (2.3) and the definition of a relative invariant. □

We have now laid all the foundations to construct a conformal generalization of the AGD bracket. This will be the main result in the section that follows.

4. A conformal analogue of the second AGD bracket

A new, purely geometrical, definition of the AGD Hamiltonian hierarchy associated with SL(n) was given in [15] by the present author. This definition was previously conjectured to be true in [10], and it can be explained as follows.

Consider the projective action of SL(n) on parametrized curves on $\mathbb{R}^{n-1}$. This action has a total of $n-1$ independent and generating differential invariants, which were discovered at the turn of the century in [21]. These invariants depend, of course, on the components of the curves and their derivatives with respect to the parameter. The formula for the most general evolution of projective curves of the form (1.1), invariant under the projective action of SL(n), was found in [10]. This evolution induces an evolution on the jet coordinates (via differentiation) and hence a parallel evolution on the generating differential invariants. It was conjectured [10] that such an evolution was indeed the AGD Hamiltonian evolution defined originally by Adler in [1], and vice versa, that any AGD evolution came from an invariant evolution of differential invariants. Furthermore, the nondegenerate matrix of relative invariants could be interpreted in this context as the Poisson tensor in '*projective coordinates*'. The result was proven to be true in [15].

In this section we will follow that same path of reasoning. We will prove that evolution (3.3) induces naturally an evolution upon the set of independent differential invariants $\{I^1, I^2\}$ found in the previous section. We will show that such an evolution is Hamiltonian, and so it can be considered as a conformal AGD bracket on the plane. At the end of the chapter we will discuss the implications of this result and a number of questions that it poses.

The following are a list of basic definitions related to infinite-dimensional Poisson brackets. Any reader needing more information could consult the last chapter of [17].

Definition 4.1. *Let $M \subset \mathbb{R} \times U$ be an open subset of the space of independent variable t and dependent variables $u = (u^1, \dots, u^n)$. The algebra of differential functions $P(t, u^{(n)})$ over M is denoted by $\mathcal{T}$. The quotient space under the image of the total divergence is the space $\mathcal{F}$ of functionals of the form*

$$\mathcal{H}(u^{(n)}) = \int H \, \mathrm{d}t \tag{4.1}$$

for a certain element $H \in \mathcal{T}$.

In all practical cases of importance t is taken to belong to either $\mathbb{R}$ or S^1 and the funtions $u(t)$ are assumed to be C^∞ (or analytic) and either vanishing at infinity or periodic (in all cases boundary conditions are imposed so that integration by parts can be used without care for the boundary terms). The elements of $\mathcal{F}$ are usually assumed to be differentiable in the Frechèt sense.

(a) Let $\mathcal{H} \in \mathcal{F}$ be defined as the Hamiltonian functional $\mathcal{H}(u^{(n)}) = \int H \, \mathrm{d}t$ for some $H \in \mathcal{T}$. We say that $\frac{\delta \mathcal{H}}{\delta u^i}$ is the Frechèt derivative of $\mathcal{H}$ in the direction of u^i whenever

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{H}(u^1, \dots, u^i + \epsilon v_i, \dots, u^n) = \int \frac{\delta \mathcal{H}}{\delta u^i}(t, u^{(n)}) v_i \, \mathrm{d}t.$$

(b) The vector function $\delta\mathcal{H} = (\frac{\delta \mathcal{H}}{\delta u^1}, \frac{\delta \mathcal{H}}{\delta u^2}, \dots, \frac{\delta \mathcal{H}}{\delta u^n}) \in \mathcal{T} \times \cdots \times \mathcal{T} = \mathcal{T}^n$ is called the functional gradient of $\mathcal{H}$.

(c) Let $\mathcal{D} : \mathcal{T}^n \to \mathcal{T}^n$ be an $n \times n$ matrix whose entries are linear differential operators. Define the following bilinear operation:

$$\begin{aligned} &\{,\} : \quad \mathcal{F} \times \mathcal{F} \to \mathcal{F} \\ &\{\mathcal{G}, \mathcal{H}\}(u^{(n)}) = \int \delta\mathcal{G} \mathcal{D} \delta\mathcal{H} \, \mathrm{d}t. \end{aligned} \tag{4.2}$$

We say that $\mathcal{D}$ is Hamiltonian and that $\{,\}$ is a Poisson bracket whenever $\{,\}$ holds the following two properties:

(i) $\{\mathcal{G}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{G}\}$

(ii) $\{\mathcal{P}, \{\mathcal{G}, \mathcal{H}\}\} + \{\mathcal{G}, \{\mathcal{H}, \mathcal{P}\}\} + \{\mathcal{H}, \{\mathcal{P}, \mathcal{G}\}\} = 0$ (Jacobi's identity)

for all functionals $\mathcal{P}, \mathcal{G}, \mathcal{H} \in \mathcal{F}$. Note that the so-called Leibniz's property for Poisson brackets immediately holds true once we ask $\mathcal{D}$ to have linear differential operators as entries.

Theorem 4.2. *Let I^1 and I^2 be defined as in theorem 2.17. Then, if the family of curves $c(s,t) = (u^1, u^2)(s,t)$ satisfies $|c_t| \neq 0$ and evolves according to (3.3), I^1, I^2 evolve according to the following evolution:*

$$\begin{pmatrix} I^1 \\ I^2 \end{pmatrix}_s = \begin{pmatrix} \partial^3 + 2I^2\partial + I^2_t & 2I^1\partial + I^1_t \\ 2I^1\partial + I^1_t & -\partial^3 - 2I^2\partial - I^2_t \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \tag{4.3}$$

where J_1, J_2 are the same differential invariants as appear in (3.3) and where $\partial = \frac{\mathrm{d}}{\mathrm{d}t}$.

Furthermore, if $\mathcal{H} \in \mathcal{F}$ and (J_1, J_2) coincide with its functional gradient,

$$(J_1, J_2) = \left(\frac{\delta\mathcal{H}}{\delta u^1}, \frac{\delta\mathcal{H}}{\delta u^2}\right)$$

then (4.3) is Hamiltonian and $\mathcal{H}$ is its associated Hamiltonian functional.

Notice that the condition imposed upon (J_1, J_2) in the theorem coincides with the one imposed on the general invariant vector in [10].

Proof. The first part of the theorem can easily be proved straightforwardly. It suffices to apply the total derivative with respect to s to the invariants (I^1, I^2), to use the fact that s and t differentiation commute, and to make use of formula (3.3) to rewrite the I-evolution as it is shown in the statement of the theorem.

The second part is more involved. We need to show that if $\{,\}$ is the bracket defined by $\mathcal{D}$ via (4.2), where $\mathcal{D}$ is given by

$$\mathcal{D} = \begin{pmatrix} \partial^3 + 2I^2\partial + I^2_t & 2I^1\partial + I^1_t \\ 2I^1\partial + I^1_t & -\partial^3 - 2I^2\partial - I^2_t \end{pmatrix} \tag{4.4}$$

then $\{,\}$ is a Poisson bracket.

Some of these properties are quite obvious. For example (4.4) is obviously bilinear and it satisfies Leibniz's rule, since the entries of $\mathcal{D}$ are all linear differential operators. It is also clear that $\mathcal{D}^* = -\mathcal{D}$ and so $\{\mathcal{G}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{G}\}$. Therefore we will focus on the Jacobi identity.

Our aim is to show that

$$\{\mathcal{P}, \{\mathcal{G}, \mathcal{H}\}\}(I_1^{(n)}, I_2^{(n)}) + \{\mathcal{G}, \{\mathcal{H}, \mathcal{P}\}\}(I_1^{(n)}, I_2^{(n)}) + \{\mathcal{H}, \{\mathcal{P}, \mathcal{G}\}\}(I_1^{(n)}, I_2^{(n)}) = 0 \tag{4.5}$$

for any functionals $\mathcal{P}, \mathcal{G}, \mathcal{H} \in \mathcal{F}$. Denote by P, G and H the kernels associated with $\mathcal{P}$, $\mathcal{G}$ and $\mathcal{H}$ respectively, so that $\mathcal{P}(I) = \int P \, \mathrm{d}t$, etc, as it is shown in (4.1). Then, we need to show that the kernel associated with the functional shown in (4.5) is the total derivative with respect to t of a certain differentiable function depending on t, I_1, I_2 and their derivatives with respect to t. Then, the functional itself will be zero as an element of $\mathcal{F}$. Since $\mathcal{D}$ has linear differential operators as entries (that is $\{,\}$ follows Leibniz's rule), we can assume that $\mathcal{P}$, $\mathcal{G}$ and $\mathcal{H}$ are linear functionals. That is, their gradients will be vectors independent of I_1, I_2 and their derivatives.

Finally, for the sake of simplicity, let us call $\frac{\delta\mathcal{H}}{\delta u^1} = h_1$ and $\frac{\delta\mathcal{H}}{\delta u^2} = h_2$, and let us assume similar notation for $\mathcal{G}$ and $\mathcal{P}$. Then, we can rewrite

$$\{\mathcal{G}, \mathcal{H}\}(I_1^{(1)}, I_2^{(1)}) = \int (g_1 g_2) \mathcal{D} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \mathrm{d}t.$$

From the expression above, we learn that the gradient of $\{\mathcal{G}, \mathcal{H}\}$ is given by

$$\begin{aligned} &(2g_1h_1' - (g_1h_1)' - 2g_2h_2' + (g_2h_2)' \qquad 2g_1h_2' - (g_1h_2)' + 2g_2h_1' - (g_2h_1)') \\ &\quad = \left(\begin{vmatrix} g_1 & h_1 \\ g_1' & h_1' \end{vmatrix} - \begin{vmatrix} g_2 & h_2 \\ g_2' & h_2' \end{vmatrix}, \begin{vmatrix} g_1 & h_1 \\ g_2' & h_2' \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ g_1' & h_1' \end{vmatrix} \right) \end{aligned}$$

where ' (also!) denotes differentiation with respect to t. Therefore, the kernel associated with $\{\mathcal{P}, \{\mathcal{G}, \mathcal{H}\}\}(I_1^{(1)}, I_2^{(1)})$ is given by

$$(p_1 p_2)\mathcal{D}(I_1^{(1)}, I_2^{(1)}) \begin{pmatrix} \begin{vmatrix} g_1 & h_1 \\ g_1' & h_1' \end{vmatrix} - \begin{vmatrix} g_2 & h_2 \\ g_2' & h_2' \end{vmatrix} \\ \begin{vmatrix} g_1 & h_1 \\ g_2' & h_2' \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ g_1' & h_1' \end{vmatrix} \end{pmatrix}.$$

If we denote by $\mathcal{Z}(I) = \partial^3 + 2I\partial + I_t$ and $\mathcal{S}(I) = 2I\partial + I_t = \partial I + I\partial$, we should recall at this point that these are well known Hamiltonian operators, the first one being the second Hamiltonian structure for the KdV equation. That is, the brackets defined on the set of functionals in one variable I and their derivatives with respect to t as

$$\{\mathcal{L}, \mathcal{R}\}_1(I^{(n)}) = \int l\mathcal{S}(I)r \, \mathrm{d}t$$

$$\{\mathcal{L}, \mathcal{R}\}_1(I^{(n)}) = \int l\mathcal{Z}(I)r \, \mathrm{d}t$$

both obey Jacobi's identity (here $l = \frac{\delta\mathcal{L}}{\delta I}$ and $r = \frac{\delta\mathcal{R}}{\delta I}$). This implies the following: note that the gradient of $\{\mathcal{L}, \mathcal{R}\}_1(I)$ and $\{\mathcal{L}, \mathcal{R}\}_2(I)$ are both identically equal to

$$2lr' - (lr)' = \begin{vmatrix} l & r \\ l' & r' \end{vmatrix}$$

and so Jacobi's identity implies that the cyclic sums (in $(\mathcal{B}, \mathcal{L}, \mathcal{R})$) of both

$$\begin{aligned} \{\mathcal{B}, \{\mathcal{L}, \mathcal{R}\}_1, \}_1(I^{(1)}) &= b\mathcal{S}(I) \begin{vmatrix} l & r \\ l' & r' \end{vmatrix} \\ \{\mathcal{B}, \{\mathcal{L}, \mathcal{R}\}_2, \}_2(I^{(1)}) &= b\mathcal{Z}(I) \begin{vmatrix} l & r \\ l' & r' \end{vmatrix} \end{aligned} \tag{4.6}$$

are equal to zero.

Going back to our original bracket, we need to show that the cyclic sum (in $(\mathcal{P}, \mathcal{G}, \mathcal{H})$) of

$$p_1 \left[\mathcal{S}(I_1) \left(\begin{vmatrix} g_1 & h_1 \\ g_1' & h_1' \end{vmatrix} - \begin{vmatrix} g_2 & h_2 \\ g_2' & h_2' \end{vmatrix} \right) + \mathcal{Z}(I_2) \left(\begin{vmatrix} g_1 & h_1 \\ g_2' & h_2' \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ g_1' & h_1' \end{vmatrix} \right) \right]$$
$$+ p_2 \left[\mathcal{Z}(I_2) \left(\begin{vmatrix} g_1 & h_1 \\ g_1' & h_1' \end{vmatrix} - \begin{vmatrix} g_2 & h_2 \\ g_2' & h_2' \end{vmatrix} \right) - \mathcal{S}(I_1) \left(\begin{vmatrix} g_1 & h_1 \\ g_2' & h_2' \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ g_1' & h_1' \end{vmatrix} \right) \right]$$

is the total derivative of some function depending on I_1, I_2 and their derivatives. This expression can be reorganized as

$$p_1\mathcal{S}(I_1) \begin{vmatrix} g_1 & h_1 \\ g_1' & h_1' \end{vmatrix} - p_2\mathcal{Z}(I_2) \begin{vmatrix} g_2 & h_2 \\ g_2' & h_2' \end{vmatrix} \tag{4.7}$$

$$+ p_1\mathcal{Z}(I_2) \begin{vmatrix} g_1 & h_1 \\ g_2' & h_2' \end{vmatrix} + p_1\mathcal{Z}(I_2) \begin{vmatrix} g_2 & h_2 \\ g_1' & h_1' \end{vmatrix} + p_2\mathcal{Z}(I_2) \begin{vmatrix} g_1 & h_1 \\ g_1' & h_1' \end{vmatrix} \tag{4.8}$$

$$- p_1\mathcal{S}(I_1) \begin{vmatrix} g_2 & h_2 \\ g_2' & h_2' \end{vmatrix} - p_2\mathcal{S}(I_1) \begin{vmatrix} g_2 & h_2 \\ g_1' & h_1' \end{vmatrix} - p_2\mathcal{S}(I_1) \begin{vmatrix} g_1 & h_1 \\ g_2' & h_2' \end{vmatrix}. \tag{4.9}$$

The cyclic sum of the first two terms is equal to a total derivative given that both $\{,\}_1$ and $\{,\}_2$ obey Jacobi's identity. The cyclic sum of the three terms in (4.8) reorganizes as

$$p_1\mathcal{Z}(I_2) \begin{vmatrix} g_1 & h_1 \\ g_2' & h_2' \end{vmatrix} + g_1\mathcal{Z}(I_2) \begin{vmatrix} h_2 & p_2 \\ h_1' & p_1' \end{vmatrix} + h_2\mathcal{Z}(I_2) \begin{vmatrix} p_1 & g_1 \\ p_1' & g_1' \end{vmatrix}$$
$$+ g_1\mathcal{Z}(I_2) \begin{vmatrix} h_1 & p_1 \\ h_2' & p_2' \end{vmatrix} + h_1\mathcal{Z}(I_2) \begin{vmatrix} p_2 & g_2 \\ p_1' & g_1' \end{vmatrix} + p_2\mathcal{Z}(I_2) \begin{vmatrix} g_1 & h_1 \\ g_1' & h_1' \end{vmatrix}$$
$$+ h_1\mathcal{Z}(I_2) \begin{vmatrix} p_1 & g_1 \\ p_2' & g_2' \end{vmatrix} + p_1\mathcal{Z}(I_2) \begin{vmatrix} g_2 & h_2 \\ g_1' & h_1' \end{vmatrix} + g_2\mathcal{Z}(I_2) \begin{vmatrix} h_1 & p_1 \\ h_1' & p_1' \end{vmatrix}.$$

Each one of the three groups of three terms above is equal to a total derivative due to the Jacobi identity of $\{,\}_2(I_2^{(1)})$. (Note that the fact that p_i, g_i and h_i, $i = 1, 2$ are constant—independent of I_1 and I_2—is fundamental here.) The same situation occurs with the final group of three terms (4.9). You simply need to interchange 1 and 2 subindices and apply Jacobi's identity for $\mathcal{S}(I_1)$. This concludes the proof of the theorem. □

5. Conclusion

We have presented here an study showing that the invariant evolutions of conformal differential invariants for parametrized curves in $\mathbb{R}^2$ are, indeed, Hamiltonian evolutions. We have additionally found the explicit expression of a set of independent and generating differential invariants, fully classifying all differential invariants. Additionally, we have given explicitly the Poisson tensor associated with this Hamiltonian evolution.

The number of questions that this paper poses is by far larger that the number of questions it answers. To name a few, I start with the possibility of a family of brackets. In the projective case, generalizing the KdV Hamiltonian evolution to higher dimensions produced a Poisson bracket for each one of the dimensions, i.e., a Hamiltonian evolution for family of curves in $\mathbb{R}^n$. The first natural question is whether such is the case for the conformal analogue. Another interesting question is what happens in the case of $\mathrm{O}(p+1, q+1)$, $p+q=n$, or, in general, in the case of any group action. If one looks a bit closer at the case of groups whose invariants are well known (for example the Euclidean group), it becomes clear after a while that not all groups have the property described here for O(3, 1) and in [15] for the projective group. For example, it often happens that the number of generating invariants is smaller or larger that the dimension and hence it is impossible or unlikely to obtain a Hamiltonian evolution similar to the KdV kind. Therefore we would need some sort of classification of those groups for which the result holds true. This study must be done in a general setting, perhaps avoiding the explicit search for differential invariants, but not their classification. The best approach could be the use of antiplectic pairs as described by Wilson in [22] and [23]. This approach was successfully applied by the present author in the projective case [15]. Of course, ideally one should try to describe conformal differential invariants in a fashion similar to the one found in [21], but it seems to be an approach less likely to succeed, since it seems to be tailored to each special case.

Finally, it should be possible to successfully apply the method described here to the classification of differential invariants of reparametrizations of $\mathbb{R}^n$, invariant under the conformal group. Furthermore, this classification can aid also in classifying cocycles generalizing the Schwarz derivative in the conformal case (considered as a nontrivial cocycle). In this sense, generalizations of the Schwarz derivative to higher dimensions were found by Bouarroudj and Ovsienko in [2] for the projective case and it has applications to quantization.

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